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On the singularities of quadratic forms

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Abstract

The induced structures on the submanifolds in a pseudo-Riemannian manifold are not, in general, pseudo-Riemannian; also, the kernel's distributions of the induced quadratic forms do not define, in general, regular foliations. In this report, the singularities of the quadratic forms on a manifold are described in a generic context and we study their geometric and algebraic properties. Therefore, using these results, we treat the problem whether there are Lagrangians on the tangent bundle of a manifold that define a Lagrangian vector field everywhere on the tangent bundle, despite the fact that their Legendre transformation is singular, and the projection of its integral curves gives the solutions of the corresponding variational problem on the manifold.

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0. Introduction

The induced structures on the submanifolds in a pseudo-Riemannian manifold are not, in general, pseudo-Riemannian. This situation appears, for instance, in the case of compact submanifolds in $\mathbb{R}^n$ equipped with the Lorentzian metric. In the classical literature one encounters the situation that corresponds to the submanifolds M where the tangent bundle TM and its orthogonal bundle $TM^{\perp}$ are supplementary, or the distribution $T_xM \cap T_xM^{\perp}$, $x \in M$, is of constant rank and defines a regular foliation on M [1,8,11,14,17–20]. However, this condition may not be satisfied and consequently singular pseudo-Riemannian

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structures with variable fibration appear, for instance, in General Relativity and other domains [5,24,26,28,32–34].

In this report the singularities of the quadratic forms on a manifold are described in a generic context. First a complete classification of the germs of generic quadratic forms is presented and the algebraic equations that define locally their degeneracy locus are given explicitly; it is shown that this algebraic set enjoys the differentiable nullstellensatz. In a pseudo-Riemannian manifold, the submanifolds in general position inherit a quadratic form enjoying these generic properties. Therefore, we characterize the image of the quadratic morphism between tangent and cotangent vector bundles, defined through a generic quadratic form; it is shown that this image consists of the pfaffian forms that annihilate the kernel of the quadratic form at the points of the smooth part of its degeneracy locus. In the case of the generic submanifolds in a pseudo-Riemannian manifold, we prove that the pfaffian forms that belong to the image of the induced quadratic morphism extend smoothly to pfaffian forms on the ambient manifold with gradient vector field tangent to the submanifold. Furthermore, a singular connection can be used for the determination of the geodesics relatively to the induced quadratic form on the generic submanifolds. Precisely, under the additional assumption that the degeneracy locus is autoparallel, the classical connection is extended to a map which induces on every stratum the Lévi-Cività connection.

In a more general context, it appears the singular variational problem implied by a Lagrangian function on the tangent bundle of a manifold M not satisfying, for instance, the classical convexity conditions. The corresponding Legendre transformation between TM and T^*M presents a critical locus and its projection gives the singular locus of the variational problem on M. In the theory of fields one encounters these transformations in the course of the introduction of moments for the fields. However, this transformation is often singular for symmetry reasons, leading to dimensional reductions, and in the physics literature, this problem has stimulated the development of various fruitful methods, cf. [12,14]. Therefore, we pose the problem whether there are Lagrangians that define a vector field everywhere on TM despite the fact that their Legendre transformation is singular and the projection of its integral curves gives the solutions of the variational problem on M.

Using the preceding results, we treat the variational problem implied by Lagrangian functions in presence of generic singularities. Specifically, we give the conditions under which the Lagrangian vector field is extended smoothly everywhere on the tangent bundle overpassing the obstruction of the critical locus of the corresponding Legendre transformation; the projection of its integral curves gives the solutions of the singular variational problem on M. Finally, the corresponding questions are examined for the singular quadratic forms that are pullbacks of a pseudo-Riemannian metric through a generic map between equidimensional manifolds. In this context, these singular quadratic forms are not generic and we prove that the pfaffian forms that possess dual gradient vector fields are exactly those which satisfy the second order kernel condition on the smooth part of the degeneracy locus. Furthermore, a singular connection constructed according to the above approach can be used for the determination of the geodesics relatively to the pullback of the pseudo-Riemannian metric.

The above results can be used for the description of certain generic properties concerning the wave front of the fundamental solution of the second order differential operators with variable coefficients. Also, these results can be used in the study of the eventual singularities of the electrical circuits. Precisely, these equations are defined through the pullback of a pseudo-Riemannian metric associated with the electrical circuit and frequently present singularities of the type described in this paper; the evolution is defined by a gradient vector field associated through the singular quadratic morphism to specific pfaffian forms on a submanifold in the ambient pseudo-Riemannian manifold, cf. [27].

Part of these results has been annnounced in [31] and a great part could be developed in the infinite dimensional context, in particular, in the context of the Fredholm quadratic structures on a Hilbert manifold. From the geometric viewpoint there are many studies performed on the singular pseudo-Riemannian or singular symplectic structures used in this paper, cf. [2,15,16,25,29,30,35].

In the sequel, the term *quadratic structure* $(M, \mathfrak{q})$ signifies a smooth manifold M of finite dimension, equipped with a quadratic form $\mathfrak{q}$ defined on M and in the case where $\mathfrak{q}$ is non degenerate we have a classical pseudo-Riemannian structure or *regular quadratic structure*; the term smooth or differentiable signifies differentiability to infinite order.

1. Generic properties of quadratic forms

1.1. Generic quadratic forms

Let $\mathcal{M}$ be a smooth manifold of dimension n and $\pi : \mathcal{V} \to \mathcal{M}$ a vector bundle of rank m. We consider the vector bundle $\mathcal{S}^2(\mathcal{V}^*)$ consisting of the quadratic forms on $\mathcal{M}$ where $\mathcal{V}^*$ denotes the dual vector bundle of $\mathcal{V}$, the fiber type of which consists of the quadratic forms on $\mathbb{R}^m$ and the structural group $GL(m, \mathbb{R})$ operates on the fibers in the usual way. This bundle receives a coherent stratification and every stratum is a fibered connected submanifold $\sum p, q, s$ having as fiber the corresponding orbit of the action of $GL(m, \mathbb{R})$ on $\mathbb{R}^{m(m+1)/2}$ that leaves invariant the numbers of positive p, negative q and zero s eigenvalues. The lemma of transversality applies in this context asserting that in the space of sections of $\mathcal{S}^2(\mathcal{V}^*)$ those that meet transversally its stratification constitute an open and dense subset in the Whitney C^∞-topology; we call these *generic quadratic forms* on $\mathcal{M}$. If $\mathfrak{q} : \mathcal{M} \to \mathcal{S}^2(\mathcal{V}^*)$ is a generic quadratic form, the manifold $\mathcal{M}$ receives by transversality a coherent stratification where every stratum

$$\Sigma_s(\mathfrak{q}) = \{x \in \mathcal{M}/\dim \mathrm{Ker}_x \mathfrak{q} = s\}, \quad 0 \le s \le n,$$

is, if nonempty, a smooth submanifold of codimension $s(s+1)/2$, admitting a partition into the sets specified by the signature (p, q, s) for s fixed, and its closure satisfies

$$\overline{\Sigma_s(\mathfrak{q})} = \bigcup_{s' \ge s} \Sigma_{s'}(\mathfrak{q}).$$

The techniques of versal unfolding lead to a finite classification of the germs of generic quadratic forms, using, in a local coordinate system $(x_1, \dots, x_n)$ centered at the origin in $\mathbb{R}^n$, the normal quadratic form

$$\mathfrak{q}_{p,q,s}(x) = \mathcal{R}_{p,q}(x) + \mathcal{S}_s(x)$$

with

$$\mathcal{R}_{p,q}(x) = \mathfrak{v}_1^2 + \cdots + \mathfrak{v}_p^2 - \mathfrak{v}_{p+1}^2 - \cdots - \mathfrak{v}_{p+q}^2$$

and

$$\mathcal{S}_s(x) = \sum_{p+q<i\le j\le m} x_{\tau(i,j)}\mathfrak{v}_i\mathfrak{v}_j,$$

$\{\mathfrak{v}_1, \dots, \mathfrak{v}_m\}$ being a local frame of the vector bundle $\mathcal{V}^*$ and $\tau : \{(i, j) \in \mathbb{N}^2 / p + q < i \le j \le m\} \to \{1, \dots, n\}$ an arbitrary injection asserting the genericity. Precisely, the lemma of *related germs*, cf. [30], suggests that if $\mathfrak{q}$ is a germ of generic quadratic form at $a \in \Sigma_s(\mathfrak{q})$, then there exist germs

$$\varphi : (\mathbb{R}^n, 0) \to (\mathcal{M}, a) \quad \text{and} \quad \psi : (\mathbb{R}^n, 0) \to GL(m, \mathbb{R})$$

of a diffeomorphism and of a smooth map, respectively, such that in the neighborhood of $0 \in \mathbb{R}^n$, in the matricial context, holds

$$\mathfrak{q}(\varphi(x)) = {}^t\psi(x)\mathfrak{q}_{p,q,s}(x)\psi(x).$$

Clearly, the diffeomorphism φ maps locally, stratum by stratum, the stratification induced by $\mathfrak{q}$ on $\mathcal{M}$ to the stratification defined by $\mathfrak{q}_{p,q,s}$ on $\mathbb{R}^n$, and in particular, gives the possibility to calculate explicitly the local equations that define the strata of the degeneracy locus:

$$\Sigma(\mathfrak{q}) = \bigcup_{s>0} \Sigma_s(\mathfrak{q}).$$

Precisely, in an appropriate chart centered at $a \in \Sigma_s(\mathfrak{q})$, the germ of the degeneracy locus is locally diffeomorphic to the algebraic set defined by the equation $\det \mathcal{S}_s(x) = 0$; evidently, $\det \mathcal{S}_s(x)$ is a homogeneous polynomial of degree s, with $s(s-1)/2$ variables appearing with exponent at most 2 while the rest s variables appear with exponent at most 1; this polynomial is a case of a quadratic-affine polynomial.

1.2. Divisions on the degeneracy locus

A homogeneous polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$, without multiple factors, is called *quadratic-affine*, when each of the factors, if we fix all variables but one x_i, $i = 1, \dots, n$, is expressed as

$$A_i(x_1, \dots, \hat{x}_i, \dots, x_n)x_i^2 + B_i(x_1, \dots, \hat{x}_i, \dots, x_n)x_i + C_i(x_1, \dots, \hat{x}_i, \dots, x_n)$$

with $A_i, B_i, C_i \in \mathbb{R}[x_1, \ldots, \hat{x}_i, \ldots, x_n]$, $\hat{x}_i$ denoting the exclusion of the variable x_i and certain A_i vanish identically; the variables such that $A_i \equiv 0$ are called *affine* and the rest *quadratic*. The following lemma, in particular, asserts that the degeneracy locus of the generic quadratic structures $(\mathcal{M}, \mathfrak{q})$ possesses the *differentiable nullstellensatz*.

Lemma 1. *In the ring of germs of smooth functions on an n-dimensional manifold, every quadratic-affine polynomial with at least* $E[(\sqrt{1+8n} - 1)/2]$ *affine variables generates the ideal of smooth functions that vanish on the zero locus.*

Proof. We proceed by induction on the number of the indeterminates of the quadratic-affine polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$. For $n = 1$, it is evidently true since P is affine; we assume it holds for $n - 1$. If all the coordinate hyperplanes $\{x_i = 0\}$ are contained inside the zero locus $\Sigma = P^{-1}(0)$ by definition the polynomial is a multiple of $x_1 \cdots x_n$ and hence the lemma follows immediately. Suppose that there exists a coordinate hyperplane, let us say the $x_1 = 0$, that is not contained inside Σ. Then, perturbing this axis through the following linear change of coordinates $z = x_1$ and $y = (y_2, \ldots, y_2)$ where $y_i = x_i + \varepsilon_i x_1$, $\varepsilon_i > 0, i = 2, \ldots, n$, we write $\bar{P}(z, y) = P(x_1, y_2 - \varepsilon_2 x_1, \ldots, y_n - \varepsilon_n x_1)$, arranging it in decreasing powers of z:

$$\bar{P}(z, y) = \alpha_v z^v + \alpha_{v-1}(y) z^{v-1} + \cdots + \alpha_1(y) z + \alpha_0(y).$$

If the variable z, for instance, is an affine one then the coefficients do not contain the required number of affine variables. However, we could assume that there exists a quadratic variable the coordinate plane of which is contained in Σ, hence divide the polynomial and the function by this variable in order to obtain the right number of affine variables, as implied by the assumption on the minimal number of affine variables of the polynomial; in this way, the resulting coefficients are quadratic-affine polynomials. By the smooth division theorem [3], for every germ of smooth function F with $F|\Sigma = 0$, we obtain

$$F(z, y) = \bar{P}(z, y) Q(z, y) + \sum_{i=0}^{v-1} R_i(y) z^i$$

with Q, R_i germs of smooth functions. The function $R_0(y)$ vanishes on $\Sigma \cap \{z = 0\}$ and from the inductive assumption we have $R_0(y) = S(y) a_0(y)$ where S is the germ of smooth function and therefore

$$R_0(y) = S(y) \left(\bar{P}(z, y) - \alpha_v z^v - \sum_{i=0}^{v-1} \alpha_i(y) z^i \right),$$

from which it follows that

$$F(z, y) = \bar{P}(z, y) Q'(z, y) + z \left(\sum_{i=0}^{v-1} R_i'(y) z^i \right).$$

The germ F belongs to the ideal generated by $\bar{P}$ and z; we have that $F|\Sigma = 0$ and therefore on $\Sigma \setminus (\Sigma \cap \{z = 0\})$:

$$\sum_{i=0}^{r-1} R'_i(y) z^i = 0.$$

In order to show that $R'_0(y) = 0$ on $\Sigma \cap \{z = 0\}$, we consider

$$P(x_1, \ldots, x_n) = a(x_2, \ldots, x_n)x_1^2 + b(x_2, \ldots, x_n)x_1 + c(x_2, \ldots, x_n).$$

The zero locus of this polynomial may represent a line, a regular graph or generally a conic set. The situation where these are contained inside the hyperplane $\{x_1 = 0\}$ is readily handled by the inductive assumption. We restrict our attention to the general case when there is no hyperplane containing the set Σ; therefore $\Sigma \setminus (\Sigma \cap \{z = 0\})$ is dense in Σ, hence $R'_0(y)$ is zero on $\Sigma \cap \{z = 0\}$. Repeating verbally the previous argument, we deduce that F belongs to the ideal generated by $\bar{P}$ and z^2. Proceeding in this way, we obtain that F belongs to the ideal generated by $\bar{P}$ and z^k, $k \in \mathbb{N}$:

$$F \in \mathfrak{I}(\bar{P}) + \bigcap_{k \in \mathbb{N}} \mathfrak{I}((z)^k).$$

Then, the Taylor series of F at the origin belongs to the ideal generated by P in the ring of formal series. The smooth division is achieved due to a division lemma of Lojasiewicz, cf. [21,22,30], generalized further in [4], that reduces the division of a smooth function by an analytic one to the division of their formal series and this completes the proof. □

1.3. Generic embeddings in a pseudo-Riemannian manifold

Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold, $\mathcal{M}$ a manifold with $\dim \mathcal{M} \leq \dim \mathcal{N}$ and $\mathcal{C}^\infty(\mathcal{M}, \mathcal{N})$ the space of smooth mappings from $\mathcal{M}$ to $\mathcal{N}$ equipped with the Whitney $\mathcal{C}^\infty$-topology. We denote by $\mathcal{I}(\mathcal{M}, \mathcal{N})$ and $\mathcal{E}(\mathcal{M}, \mathcal{N})$, respectively, the open subsets of immersions and proper embeddings of $\mathcal{M}$ in $\mathcal{N}$; also, we denote by $\mathcal{I}^1(\mathcal{M}, \mathcal{N})$ the fibering by the 1-jets of these immersions. The morphism $j^1\mathfrak{h} \to \mathfrak{h}^*\mathfrak{g}$ of the fiberings $\mathcal{I}^1(\mathcal{M}, \mathcal{N})$ and $\mathcal{S}^2(T^*\mathcal{M})$ as we can prove, by a rather direct calculation, is a submersion and therefore lifts the stratification of $\mathcal{S}^2(T^*\mathcal{M})$ to a stratification of $\mathcal{I}^1(\mathcal{M}, \mathcal{N})$. The immersions that have their 1-jet transverse to this stratification form an open and dense set in the space $\mathcal{I}(\mathcal{M}, \mathcal{N})$ and their intersection with $\mathcal{E}(\mathcal{M}, \mathcal{N})$ defines the space of *generic embeddings* of $\mathcal{M}$ in $(\mathcal{N}, \mathfrak{g})$. A direct calculation establishes Lemma 2.

Lemma 2. *A proper embedding of a manifold $\mathcal{M}$ in a pseudo-Riemannian manifold $(\mathcal{N}, \mathfrak{g})$ is generic if and only if the induced quadratic form is generic on $\mathcal{M}$.*

In particular, in a pseudo-Riemannian manifold $(\mathcal{N}, \mathfrak{g})$, a submanifold $\mathcal{M}$ is called *generic* if its inclusion $\mathfrak{i} : \mathcal{M} \to (\mathcal{N}, \mathfrak{g})$ is a generic embedding. Therefore, we conclude that on every generic submanifold there is an induced generic quadratic form $\mathfrak{q} = \mathfrak{i}^*\mathfrak{g}$ that we described in the preceding paragraphs and then we have a generic quadratic structure $(\mathcal{M}, \mathfrak{q})$. Thus, using the results of Section 1.1, we obtain a finite classification of the germs of the generic submanifolds in a pseudo-Riemannian manifold.

2. Dynamics in the presence of generic stratified singularities

2.1. Singular quadratic duality

Let $\pi : \mathcal{V} \to \mathcal{M}$ be a vector bundle and $\mathrm{Hom}(\mathcal{V}, \mathcal{V}^*)$ the bundle of homomorphism between $\mathcal{V}$ and its dual $\mathcal{V}^*$ over the manifold $\mathcal{M}$. Every quadratic form $\mathfrak{q} : \mathcal{M} \to \mathcal{S}^2(\mathcal{V}^*)$ gives rise to a smooth section of $\mathrm{Hom}(\mathcal{V}, \mathcal{V}^*)$ that satisfies the symmetry condition and implies the *quadratic morphism* or $\mathfrak{q}$-morphism expresses the correspondence of $\mathcal{V}$-sections to $\mathcal{V}^*$-sections through the relation

$$\mathfrak{q}(\mathcal{X}, .) = \mathcal{X}^*(.).$$

If the kernel of $\mathfrak{q}$ is of constant dimension, then the image of the $\mathfrak{q}$-morphism defines a subbundle of $\mathcal{V}^*$ and every fiber of the latter consists of the elements of $\mathcal{V}^*$ that annihilate this kernel; in the case of vanishing kernel, the $\mathfrak{q}$-morphism is an isomorphism of vector bundles defining the classical quadratic duality between the sections of $\mathcal{V}$ and $\mathcal{V}^*$. However, when the dimension of the kernel varies, then the image of the $\mathfrak{q}$-morphism defines a *vector bundle of variable fibers*. We treat this problem in the context of the generic quadratic forms.

Theorem 3. *If $\mathfrak{q} : \mathcal{M} \to \mathcal{S}^2(\mathcal{V}^*)$ is a generic quadratic form, then the image of the $\mathfrak{q}$-morphism between the vector bundles $\mathcal{V}$ and $\mathcal{V}^*$ consists of those sections of $\mathcal{V}^*$ that annihilate the kernel of $\mathfrak{q}$ at the points of $\Sigma_1(\mathfrak{q})$.*

Proof. The sections of $\mathcal{V}^*$ that are contained in the image of the $\mathfrak{q}$-morphism, clearly, annihilate the kernel of $\mathfrak{q}$ on $\mathcal{M}$. Conversely, let $\mathcal{X}^*$ be a smooth section of $\mathcal{V}^*$ that annihilates the kernel of $\mathfrak{q}$ on $\Sigma_1(\mathfrak{q})$. We are placed first at a point of $\Sigma_1(\mathfrak{q})$ and in a neighborhood of it we trivialize the bundles; in a chart centered at this point, using the same notations in the matricial context, we look for the solution of the matricial equation

$$\mathfrak{q}(x)\mathcal{X}(x) = \mathcal{X}^*(x).$$

After Section 1.1, in the neighborhood of $0 \in \mathbb{R}^n$, using the normal quadratic form

$$\mathfrak{q}_{p,q,s}(x) = \mathcal{R}_{p,q}(x) + \mathcal{S}_s(x)$$

with $s = 1$ and $\mathcal{S}_1(x) = x_1 \mathfrak{v}_m^2$, we reduce to the equation

$$\mathfrak{q}_{p,q,1}(x)X(x) = X^*(x),$$

where $X(x) = \psi(x)\mathcal{X}(\varphi(x))$ and $X^*(x) = {}^t\psi(x)^{-1}X^*(\varphi(x))$. This allows us to look at the component equation

$$x_1 X_m(x) = X_m^*(x),$$

the second member of which vanishes on $\{x_1 = 0\}$ by the assumption, and after being divided by x_1, gives a smooth solution; this implies further the existence of a local smooth solution for the initial equation at the points of $\Sigma_1(\mathfrak{q})$. Standing now at an arbitrary point of $\Sigma(\mathfrak{q})$, the initial equation is expressed as

$$\det \mathfrak{q}(x)\mathcal{X}(x) = {}^{c}\mathfrak{q}(x)\mathcal{X}^*(x),$$

where ${}^{c}\mathfrak{q}(x)$ denotes the adjoint matrix of $\mathfrak{q}(x)$. This equation has a solution at the points of $\Sigma_1(\mathfrak{q})$ and therefore the second member vanishes on $\Sigma_1(\mathfrak{q})$ and by continuity on $\Sigma(\mathfrak{q})$. The coefficient of the first member is, in an appropriate coordinate system, modulo a unit, a quadratic-affine polynomial, and by Lemma 1, it divides the second member in the ring of germs of smooth functions. Therefore we get a local smooth solution, and by a partition of unity, we construct globally a smooth solution that defines a unique smooth section $\mathcal{X}$ of $\mathcal{V}$ associated to $\mathcal{X}^*$ over $\mathcal{M}$. □

Let $(\mathcal{M}, \mathfrak{q})$ be a generic quadratic structure, i.e. a smooth manifold equipped with a generic quadratic form $\mathfrak{q} : \mathcal{M} \to \mathcal{S}^2(T^*\mathcal{M})$. The sections of $T^*\mathcal{M}$ that are contained in the image of the $\mathfrak{q}$-morphism are called *admissible pfaffians* forms and their set is denoted

$$\mathcal{A}(\mathfrak{q}) = \{\mathfrak{b} \in \mathfrak{X}^*(\mathcal{M}) / \exists \mathcal{X}_{\mathfrak{b}} \in \mathfrak{X}(\mathcal{M}) : \mathfrak{q}(\mathcal{X}_{\mathfrak{b}}, .) = \mathfrak{b}(.)\},$$

where $\mathfrak{X}(\mathcal{M})$ and $\mathfrak{X}^*(\mathcal{M})$ denote, respectively, the module of smooth vector fields and the module of smooth pfaffian forms on $\mathcal{M}$. After Theorem 1, the elements of $\mathcal{A}(\mathfrak{q})$ are characterized by the *kernel's condition*:

$$\mathrm{Ker}_x \mathfrak{q} \subseteq \mathrm{Ker}_x \mathfrak{b}, \quad x \in \Sigma_1(\mathfrak{q}).$$

In particular, the above theorem affirms that: if a differential form satisfies the kernel's condition on $\Sigma_1(\mathfrak{q})$ then it is satisfied on all $\Sigma(\mathfrak{q})$. Following the terminology of Malgrange, cf. [30], the stratum $\Sigma_1(\mathfrak{q})$ is $\mathfrak{q}$-*dense* in $\mathcal{M}$: the existence of a formal solution of the equation

$$\mathfrak{q}(x)\mathcal{X}(x) = \mathfrak{b}(x)$$

at every point of $\Sigma_1(\mathfrak{q})$ implies the existence of a smooth solution on $\mathcal{M}$. If $\mathfrak{b} \in \mathcal{A}(\mathfrak{q})$, then the unique smooth vector field $\mathcal{X}_{\mathfrak{b}}$ on $\mathcal{M}$ satisfying the equation

$$\mathfrak{q}(\mathcal{X}_{\mathcal{B}}, .) = \mathfrak{b}(.)$$

is called *pfaffian gradient* of $\mathfrak{b}$. In the notations of the normal quadratic form, on a suitable chart $(U; x_1, \ldots, x_n)$, cf. Section 1.1, the admissible pfaffians forms are expressed as follows:

$$\mathfrak{b}(x) = \sum_{1 \le i \le n-s} b_i(x)\,\mathrm{d}x_i + \sum_{n-s < i \le j \le n} x_{\tau(i,j)} b_i(x)\,\mathrm{d}x_j,$$

where $b_1, \ldots, b_n$ are arbitrary smooth functions on U and the corresponding pfaffian gradients are

$$\mathcal{X}_{\mathfrak{b}}(x) = \sum_{1 \le i \le n} b_i(x)\partial/\partial x_i.$$

Furthermore, in the set $\mathcal{A}(\mathfrak{q})$ there exist pfaffian forms with pfaffian gradients tangent to the strata of the locus $\Sigma(\mathfrak{q})$. Since this locus is locally diffeomorphic to a space stratified by the action of a Lie group, the vector fields that are tangent to the strata are induced by the action of the Lie group. These pfaffian gradients, in the neighborhood of the points of every strata $\Sigma_s(\mathfrak{q})$, assume the following expression:

$$\mathcal{X}_{\mathfrak{b}}(x) = X_{\mathrm{reg}}(x) + X_{\mathrm{sing}}(x)$$

with

$$X_{\mathrm{reg}}(x) = \sum_{1\le i\le k} a_j(x)\partial/\partial x_j$$

and

$$X_{\mathrm{sing}}(x) = \sum_{1\le j\le k} (x_{\tau(i,k)}b_{kj}(x) + x_{\tau(k,j)}b_{ki}(x))\partial/\partial x_{\tau(i,j)},$$

where $k = n - s(s+1)/2$, and a_j and b_{ik} are arbitrary smooth functions on U.

2.2. *Extension to the ambient pseudo-Riemannian manifold*

Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and we consider a generic submanifold $\mathcal{M}$ equipped with the induced quadratic form $\mathfrak{q}$. Then the quadratic structure $(\mathcal{M}, \mathfrak{q})$ is stratified into the smooth submanifolds

$$\Sigma_s(\mathfrak{q}) = \{x \in \mathcal{M} / \dim(T_x\mathcal{M} \cap T_x\mathcal{M}^{\perp}) = s\}, \quad s \in \mathbb{N}^*.$$

The admissible pfaffians forms are exactly those which satisfy the kernel's condition on $\Sigma_1(\mathfrak{q})$ (cf. Lemma 2, Theorem 3). Actually here we pose the question concerning the possibility of the smooth extension of the admissible pfaffian forms to pfaffian forms on $(\mathcal{N}, \mathfrak{g})$ with pfaffian gradient tangent to $\mathcal{M}$.

Theorem 4. *Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and $\mathcal{M}$ a generic submanifold equipped with the induced quadratic form $\mathfrak{q}$. The pfaffian forms that satisfy the kernel's condition on $\Sigma_1(\mathfrak{q})$ are exactly those which may be smoothly extended to pfaffian forms on $\mathcal{N}$ with pfaffian gradient tangent to $\mathcal{M}$.*

Proof. Let $\mathfrak{b}$ be a pfaffian form on $\mathcal{M}$ that satisfies the kernel's condition on $\Sigma_1(\mathfrak{q})$. After a local trivialization of the bundles in the neighbourhood of a point $a \in \mathcal{M}$, we placed in a local chart $(U'; x_1, \dots, x_n, y_1, \dots, y_k)$ centered at this point and the defining equations of $U' \cap \mathcal{M} = U$ are $y_1 = \cdots = y_k = 0$. The vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ define the tangent bundle TU and the vector fields $\partial/\partial y_1, \dots, \partial/\partial y_k$ define a subbundle N complementary to TU in TU'; also, through the $\mathfrak{g}$-morphism, the dual vector fields $Y_1, \dots, Y_k$ of $\mathrm{d}y_1, \dots, \mathrm{d}y_k$ generate a subbundle $TU^{\perp}$ in TV. If $\Pi : TU^{\perp} \to N$ is the projection parallel to TU, we have that $\Pi = (\pi_{ij})$ with $\pi_{ij} = \mathfrak{g}(Y_i, Yj)$, and for every $x \in U$,

$$\mathrm{Ker}_x\Pi = T_xU \cap T_xU^{\perp} = \mathrm{Ker}_x\mathfrak{q}.$$

The existence of a smooth extension $\mathfrak{b}'$ of $\mathfrak{b}$ on U':

$$\mathfrak{b}' = \mathfrak{b} + \sum_{i=1}^{k} \varrho_i \, \mathrm{d}y_i$$

with pfaffian gradient $\mathcal{X}_{\mathfrak{b}'}$ tangent to U is equivalent to the existence of a system of coordinates on U' and $\varrho = (\varrho_1, \dots, \varrho_k)$, $\varrho_i \in C^{\infty}(U)$ satisfying the equations

$$\sum_{i=1}^{k} \pi_{ij} \varrho_i = -\mathfrak{b}'(X_j),$$

where X_j is the component of Y_j inside TU, $j = 1, \dots, k$, cf. [29]. This system could be written as

$$(\det \Pi)\beta = -{}^{c}\Pi(\mathfrak{b}(X)),$$

where $\mathfrak{b}(X) = (\mathfrak{b}(X_1), \dots, \mathfrak{b}(X_k))$ and its clear that $\det \Pi$ and $\det \mathfrak{q}$ generate the same ideal in the ring of smooth functions on U. The genericity of $\mathfrak{q}$ implies the $\mathfrak{q}$-density and hence the Π-density of $\Sigma_1(\mathfrak{q})$ in $\mathcal{M}$, cf. Section 2.1. Then, along the lines of Theorem 3, we prove that this system has a formal solution in the neighborhood of a point of $\Sigma_1(\mathfrak{q})$ and we conclude that the existence of a local smooth solution on $\mathcal{M}$ and hence the desired local smooth extension $\mathfrak{b}'$ on $\mathcal{N}$. Therefore, by a partition of unity, we obtain a global smooth extension $\mathfrak{b}'$ on $\mathcal{N}$, unique modulo the ideal of the functions that vanish on $\mathcal{M}$, such that the pfaffian gradient $\mathcal{X}_{\mathfrak{b}'}$ is tangent to $\mathcal{M}$. Conversely, the pfaffian forms on $\mathcal{M}$ that admit a smooth extension to $\mathcal{N}$ with pfaffian gradient tangent to $\mathcal{M}$ clearly satisfy the kernel's condition on $\Sigma(\mathfrak{q})$. □

2.3. Singular connection on the degeneracy locus

Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and we consider a generic submanifold $\mathcal{M}$ equipped with the induced quadratic form $\mathfrak{q}$. On the complement of the degeneracy locus $\Sigma(\mathfrak{q})$ of the singular pseudo-Riemannian structure $(\mathcal{M}, \mathfrak{q})$ there exists a unique connection of vanishing torsion associated with $\mathfrak{q}$, it is the Lévi-Cività connection ∇. This connection coincides with the reciprocal image through the $\mathfrak{q}$-morphism of the pfaffian form $\Delta_X Y$ given by the map

$$\Delta : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}^*(\mathcal{M})$$

with

$$\Delta_X Y(Z) = \tfrac{1}{2}(X(\mathfrak{q}(Y, Z)) + Y(\mathfrak{q}(X, Z)) - Z(\mathfrak{q}(X, Y)) + \mathfrak{q}([X, Y], Z) \\ -\mathfrak{q}([X, Z], Y) - \mathfrak{q}([Y, Z], X)).$$

Let $\mathfrak{X}_{\mathfrak{q}}(\mathcal{M})$ be the set of smooth vector fields that are tangent to the strata of $\Sigma(\mathfrak{q})$ and we assume, in addition, that the kernel of $\mathfrak{q}$ is transverse to the tangent space of $\Sigma_1(\mathfrak{q})$ at every point of an open and dense set $\Sigma_1'(\mathfrak{q})$ contained in $\Sigma_1(\mathfrak{q})$. The locus $\Sigma(\mathfrak{q})$ is called *autoparallel* if the following condition holds: if K is a vector field that generates the kernel of $\mathfrak{q}$ at the points of $\Sigma_1'(\mathfrak{q})$, the Lie derivative $L_K\mathfrak{q}(X, Y)$ vanishes on $\Sigma_1'(\mathfrak{q})$ for all $X, Y \in \mathfrak{X}_\mathfrak{q}(M)$. Following the study developed in [16,25], and after Theorems 3 and 4, we deduce the result.

Theorem 5. *Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and $\mathcal{M}$ a generic submanifold and we assume that the kernel of the induced quadratic form $\mathfrak{q}$ is transverse to the tangent space of $\Sigma_1(\mathfrak{q})$ at every point of an open and dense set contained in $\Sigma_1(\mathfrak{q})$. Then, the Levi-Cività connection associated with $\mathfrak{q}$ on the complement of $\Sigma(\mathfrak{q})$ extends to a smooth map*

$$\tilde{\nabla} : \mathfrak{X}_\mathfrak{q}(\mathcal{M}) \times \mathfrak{X}_\mathfrak{q}(\mathcal{M}) \to \mathfrak{X}_\mathfrak{q}(\mathcal{M}),$$

if and only if $\Sigma(\mathfrak{q})$ is autoparallel, and in this case, on every stratum $\Sigma_s(\mathfrak{q})$, the quadratic form $\mathfrak{q}$ induces a pseudo-Riemannian metric $\mathfrak{q}_s$ and the map $\tilde{\nabla}$ induces the Lévi-Cività connection associated with $\mathfrak{q}_s$, $s \in \mathbb{N}$.

The classical situation concerning the geodesics on the ambient pseudo-Riemannian manifold $(\mathcal{N}, \mathfrak{g})$ can be generalized on the quadratic structure $(\mathcal{M}, \mathfrak{q})$ as follows: a curve $\gamma : [\alpha, \beta] \to \mathcal{M}$ is a geodesic relative to $\mathfrak{q}$ if and only if $\Delta_{\dot\gamma}\dot\gamma = 0$; in this aim it is sufficient to choose through a vector field X a local extension of $\dot\gamma$ and verify that the value of the vector field $\Delta_X X$ along γ does not depend on this choice extension. Clearly, when the curve γ is contained in the complement of $\Sigma(\mathfrak{q})$, $\Delta_{\dot\gamma}\dot\gamma = 0$ if and only if $\nabla_{\dot\gamma}\dot\gamma = 0$. Also, for $s \neq 0$, if $x \in \Sigma_s(\mathfrak{q})$ and $\mathfrak{v} \in \ker_x \mathfrak{q}$, there exists a unique local geodesic tangent to $\mathfrak{v}$ at x contained in $\Sigma_s(\mathfrak{q})$; furthermore, when we are placed at a point $\Sigma_s(\mathfrak{q})$ the geodesics contained in the complement of $\Sigma(\mathfrak{q})$ arrive tangentially to $\Sigma_1(\mathfrak{q})$ and the other geodesics remain within $\Sigma_s(\mathfrak{q})$, cf. [25].

3. The variational problem in the presence of generic stratified singularities

3.1. Singular Legendre transforms

The classical variational problem on the space of smooth curves $\gamma : [a, b] \to \mathcal{M}$ on a manifold $\mathcal{M}$, defined by a Lagrangian $L : T\mathcal{M} \to \mathbb{R}$, deals with the extrema of the functional

$$\mathcal{F}(\gamma) = \int_a^b L(x(\gamma(t)), v(\gamma(t)))\, \mathrm{d}t$$

with (x, v) the usual position–velocity coordinates. This question can be expressed as defining a smooth vector field $\mathcal{X}_L$ on $T\mathcal{M}$, called *Lagrangian vector field*, with integral curves that project exactly on the solutions of the variational problem on $\mathcal{M}$. The Lagrangian allows us to define the Legendre transformation

$$\mathcal{L} : T\mathcal{M} \to T^*\mathcal{M}, \quad \mathcal{L}(x, v) = (x, \mathrm{d}_v L(x, v)).$$

The Lagrangian vector field $\mathcal{X}_L$ is given by the equation

$$(\mathcal{L}^*\Omega)(\mathcal{X}_L, .) = -\,d\mathcal{E}_L(.),$$

where $\mathcal{L}^*\Omega$ is the Lagrangian form as the pullback on $T\mathcal{M}$ of the canonical symplectic form Ω on $T^*\mathcal{M}$ and $\mathcal{E}_L$ is the corresponding energy function on $T\mathcal{M}$. Evidently, the Lagrangian vector field is well defined out of the *degeneracy locus*

$$\Sigma(\mathcal{L}^*\Omega) = \{(x, v) \in T\mathcal{M}/\mathrm{Ker}_{(x,v)}\mathcal{L}^*\Omega \neq \{0\}\}$$

that coincides with the *critical locus:*

$$\Sigma(\mathcal{L}) = \{(x, v) \in T\mathcal{M}/\mathrm{Ker}_{(x,v)}D\mathcal{L} \neq \{0\}\}.$$

The projection of the critical locus constitutes the *singular locus* of the variational problem on $\mathcal{M}$. In the particular case where the energy function is defined through a quadratic form $\mathfrak{q}$ on $\mathcal{M}$, the singular locus of the variational problem is exactly the degeneracy locus $\Sigma(\mathfrak{q})$, cf. Section 1.1. Hence, we aim to determine, in the generic context, the conditions under which the Lagrangian vector field admits a smooth extension everywhere on $T\mathcal{M}$, and then, to describe the behavior of the solutions on the singular locus.

3.2. Generic Legendre sections

Let $\mathrm{Hom}(\pi_1^*T(T\mathcal{M}), \pi_2^*T(T^*\mathcal{M}))$ be the vector bundle of homomorphisms over the product $T\mathcal{M} \times T^*\mathcal{M}$, where π_1 and π_2 are the projections of this product on the first and second factor, respectively, over the manifold $\mathcal{M}$, equipped with the usual action of the structural group $GL(2n, \mathbb{R}) \times GL(2n, \mathbb{R})$. We are going to construct a subbundle $\Lambda(T\mathcal{M})$ called *Legendre bundle*. We recall the splitting $\mathbb{R}^{2n} = E_1 \oplus E_2$ in the subspaces $E_1 = \langle e_1, \dots, e_n\rangle$ and $E_2 = \langle e_{n+1}, \dots, e_{2n}\rangle$ with $\{e_1, \dots, e_{2n}\}$ the canonical basis and $p_i : \mathbb{R}^{2n} \to E_i$, $i = 1, 2$, the corresponding canonical projections. The typical fiber of this subbundle, with the obvious identifications, is the semialgebraic submanifold

$$\Lambda = \{\mathfrak{l} \in \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})/(p_1 \circ \mathfrak{l}|E_1) \in GL(n, \mathbb{R}), \mathfrak{l}(E_2) \subset E_2, \mathfrak{l}|E_2 = {}^t(\mathfrak{l}|E_2)\}$$

in the typical fiber of $\mathrm{Hom}(\pi_1^*T(T\mathcal{M}), \pi_2^*T(T^*\mathcal{M}))$, equipped with the reduced action, define through the linear and symplectic group by left and right multiplication, of the subgroup

$$\mathcal{G} = \{(g, g') \in GL(2n, \mathbb{R}) \times Sp(2n, \mathbb{R})/g(E_2) \subset E_2, g'(E_2) \subset E_2, \\ g'|E_2 = {}^t(g|E_2)^{-1}\}.$$

This action leaves invariant the decomposition of the elements of Λ in four blocks; the one corresponding to the symmetric operator is called *quadratic block*. Hence, the submanifold Λ is fibered over the space of the quadratic forms on $\mathbb{R}^n$ and a direct calculation allows us to verify that this action projects onto the usual action of $GL(n, \mathbb{R})$ on the space of quadratic forms. Therefore, the stratification of the space of quadratic forms lifts to a coherent stratification of this fiber and the action of the structural group carries over this stratification to the fiber space; that is the quadratic stratification of the bundle $\Lambda(T\mathcal{M})$. The smooth sections of this bundle are called *Legendre sections*; in particular, the differential of a Legendre transform defines a section of this bundle and its hessian with respect to the velocities corresponds to the quadratic block in the decomposition of the fiber type of $\Lambda(T\mathcal{M})$. In the space of Legendre sections those that are transverse to the quadratic stratification constitute an open and dense set for the Whitney $\mathcal{C}^\infty$-topology. Clearly, a Legendre section is transverse to the quadratic stratification if and only if the quadratic form corresponding to the quadratic block of this section is generic. Using the normal quadratic form $\mathfrak{q}_{p,q,s}$ given in Section 1.1, it can be constructed in an obvious way a *normal Legendre section* $\mathcal{L}_s$ on $\mathbb{R}^{2n}$ with the matricial expression

$$\mathcal{L}_s(x) = \begin{pmatrix} I_n & 0 \\ C(x) & Q_s(x) \end{pmatrix} \text{ with } Q_s(x) = \begin{pmatrix} I_{n-s} & 0 \\ 0 & S_s(x) \end{pmatrix}$$

and $C(x)$ a suitable matrix, whose precise expression will not intervene to our study, cf. Section 1.1. Hence, if $\mathcal{L}$ is a Legendre section transverse to the quadratic stratification of $\Lambda(T\mathcal{M})$ and $a \in T\mathcal{M}$ where $\dim \operatorname{Ker} \mathcal{L}(a) = s$, then there exist germs

$$\varphi : (\mathbb{R}^{2n}, 0) \to (T\mathcal{M}, a) \quad \text{and} \quad (\psi, \psi') : (\mathbb{R}^{2n}, 0) \to (\mathcal{G}, \mathrm{id})$$

of a diffeomorphism and smooth maps, respectively, such that

$$\mathcal{L}(\varphi(x)) = \psi'(x)\mathcal{L}_s(x)\psi(x).$$

Furthermore, it is possible to construct a refinement of the quadratic stratification of $\Lambda(T\mathcal{M})$ that encounters the influence of the elements of each fiber Λ on the exterior symplectic form ω of $\mathbb{R}^{2n}$, i.e. the behavior of $\mathfrak{l}^*\omega$ for $\mathfrak{l} \in \Lambda$. The novel strata

$$\Lambda_{s,c} = \{\mathfrak{l} \in \Lambda / \dim \ker \mathfrak{l} = s, \dim \ker(\omega | \operatorname{Im} \mathfrak{l}) = c\}, \quad s, c \in \mathbb{N},$$

are smooth submanifolds and codim $\Lambda_{s,c} \geq s(s+1)/2$; in particular, $\Lambda_{0,0}$ is open and dense in Λ and its complement is a singular hypersurface of which the smooth part contains the open and dense set $\Lambda_{1,1}$ cf. [27]. This stratification gives rise, by the action of the structural group to a stratification of the bundle $\Lambda(T\mathcal{M})$ called the *Legendre stratification*. The Legendre sections that are transverse to the Legendre stratification constitute an open and dense set in the space of all Legendre sections; these are called *generic Legendre sections*. In particular, the Legendre transforms with their differential transverse to this stratification are called *generic Legendre transforms*.

3.3. *Extension on the critical locus*

Actually here we proceed, in the generic context, to characterize the Lagrangians with their Lagrangian vector field extending smoothly to the whole tangent bundle eliminating the obstruction of the critical locus of the Legendre transformation. Always, $\mathcal{L}^*\Omega$ denotes the Lagrangian form and $\mathcal{E}_\mathrm{L}$ the energy function implied by the Lagrangian L on the tangent bundle of the manifold $\mathcal{M}$.

Theorem 6. *Let $L : T\mathcal{M} \to \mathbb{R}$ be a Lagrangian with generic Legendre transform $\mathcal{L} : T\mathcal{M} \to T^*\mathcal{M}$ over an n-dimensional manifold $\mathcal{M}$:*

1. *The critical locus $\Sigma(\mathcal{L})$ is a stratifed hypersurface in smooth submanifolds*

$$\Sigma_{s,c}(\mathcal{L}) = \{(x, v) \in T\mathcal{M}/\dim \mathrm{Ker}_{(x,v)} D\mathcal{L} = s, \dim \mathrm{Ker}_{(x,v)} \mathcal{L}^*\Omega = c\}, \quad s, c \in \mathbb{N}^*$$

such that

$$\overline{\Sigma_{s,c}(\mathcal{L})} = \bigcup_{i,j \in \mathbb{N}} \Sigma_{s+i,c+j}(\mathcal{L}).$$

2. *The Lagrangian vector field $\mathcal{X}_{\mathrm{L}}$ admits a smooth extension everywhere on $T\mathcal{M}$ if and only if at every point of $\Sigma_{1,1}(\mathcal{L})$ the energy function satisfies the following condition:*

$$\mathrm{d}\mathcal{E}_L \wedge (\mathcal{L}^*\Omega)^{n-1} = 0.$$

3. *The projection of the integral curves of the extended Lagrangian vector field $\mathcal{X}_{\mathrm{L}}$ give the solutions of the variational problem implied by the Lagrangian L on $\mathcal{M}$.*

Proof. The above properties of the critical locus are asserted by transversality to the Legendre stratification of the bundle $\Lambda(T\mathcal{M})$. In particular, we observe that the stratum $\Sigma_{1,1}(\mathcal{L})$ is open and dense in the critical locus $\Sigma(\mathcal{L})$. Clearly, the condition of the theorem equivels to the annhihilation of the kernel of $\mathcal{L}^*\Omega$ by the differential of $\mathcal{E}_{\mathrm{L}}$ at the points of $\Sigma_{1,1}(\mathcal{L})$, while the necessity for the smooth extension of the Lagrangian vector field is evident. Conversely, let $\mathcal{X}_{\mathrm{L}}$ be the Lagrangian vector field defined out of $\Sigma(\mathcal{L})$ and we search its smooth extension everywhere on $T\mathcal{M}$ given by the equation

$$(\mathcal{L}^*\Omega)(\mathcal{X}_{\mathrm{L}}, .) = -\mathrm{d}\mathcal{E}_{\mathrm{L}}(.).$$

We are placed at a point $a \in \Sigma_{1,1}(\mathcal{L})$. In a neighborhood of this point we trivialize the bundles, and in the local context, using the corresponding matricial notations, we look at the matricial equation

$$(\mathcal{L}^*\Omega)(x)\mathcal{X}_{\mathrm{L}}(x) = \mathcal{B}(x).$$

The transversality to the Legendre stratification asserts the transversality to the quadratic stratification of $\Lambda(T\mathcal{M})$, and after Section 3.2, using the normal Legendre section $\boldsymbol{\mathcal{L}}_s$ with $s = 1$ in the neighborhood of $0 \in \mathbb{R}^{2n}$, we may write this equation as

$${}^t\boldsymbol{\mathcal{L}}_s(x)\Omega(\mathcal{L}o\varphi(x))\boldsymbol{\mathcal{L}}_s(x)X(x) = B(x)$$

with

$$X(x) = \mathfrak{g}(x)(D\varphi^{-1})(\varphi(x))\mathfrak{X}_f(\varphi(x)) \text{ and } B(x) = {}^t\mathfrak{g}(x)((D\varphi^{-1})(\varphi(x)))^{-1}\mathcal{B}(\varphi(x)).$$

The matricial coefficient of this equation assumes the following writing:

$$\begin{pmatrix} A(x) & Q_1(x) \\ -{}^tQ_1(x) & 0 \end{pmatrix}$$

with $A(x) = C(x) - {}^{t}C(x)$ being a smooth field of $n \times n$ antisymetric matrices, and therefore, we are led to the following system of equations:

$$A(x)X_1(x) + Q_1(x)X_2(x) = B_1(x) \quad \text{and} \quad Q_1(x)X_1(x) = -B_2(x),$$

where X_i and $B_i : (\mathbb{R}^{2n}, 0) \to E_i$ are the component functions of X and B, respectively, for the subspaces $E_i, i = 1, 2$, in the canonical splitting of $\mathbb{R}^{2n}$, cf. Section 3.2. The condition of the theorem implies the annihilation of the kernel of $\mathcal{L}^*\Omega$ by the differential of $\mathcal{E}_{\rm L}$ at the points of $\Sigma_{1,1}(\mathcal{L})$; this kernel, in the neighborhood of a, computed through the normal Legendre section is spanned by the vector fields e_{2n} and $e_n - \alpha(x)e_{2n}$, where $\alpha(x) = \mathrm{Id}\, E_2 \otimes A(x)$. Consequently, there exists a formal solution for the above system of equations and then the Hadamard's lemma implies the existence of a local smooth solution on $\Sigma_{1,1}(\mathcal{L})$. Standing now at an arbitrary point of $\Sigma(\mathcal{L})$, the initial matricial equation is expressed as

$$\det(\mathcal{L}^*\Omega)(x)\mathcal{X}_L(x) = {}^{c}(\mathcal{L}^*\Omega)(x)\mathcal{B}(x).$$

The above study ensures the existence of a solution for this equation on $\Sigma_{1,1}(\mathcal{L})$ and that implies the vanishing of the second member on $\Sigma_{1,1}(\mathcal{L})$ and by continuity on $\Sigma(\mathcal{L})$. After Section 3.2, the quadratic stratification of $\Sigma(\mathcal{L})$ is mapped locally by a difeomorphism to the stratification implied by the normal Legendre section $\mathcal{L}_s$ on $\mathbb{R}^{2n}$. Hence, a direct calculation on the normal Legendre section gives the local equation of $\Sigma(\mathcal{L})$ and this is defined in an appropriate chart by a quadratic-affine polynomial. Then, in suitable coordinates, $\det(\mathcal{L}^*\Omega)(x)$ is the square of a quadratic-affine polynomial, and after Lemma 1, this polynomial divides the second member of the equation in the ring of germs of smooth functions; repeating the division process, we obtain a local smooth solution on $\Sigma(\mathcal{L})$. By a partition of unity, we construct globally a smooth solution that defines the Lagrangian vector field $\mathcal{X}_L$ on $T\mathcal{M}$ and clearly the projection of its integral curves gives the solutions of the variational problem implied by the Lagrangian L on $\mathcal{M}$. □

This situation is illustrated in the following example which could be seen as a generalization of the celebrated Weierstrass example of a variational problem having no trivial extrema, cf. [10]. Let $L : (\mathbb{R}^4, 0) \to \mathbb{R}$ be the germ of a Lagrangian on the tangent bundle of $\mathbb{R}^2$ and $\mathcal{L} : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)^*$ be the Legendre transformation that defines. The coordinates in $(\mathbb{R}^4, 0)$ are denoted by (x, x', ξ, ξ') and we assume that

$$\mathrm{d}L(0) = 0,\ \partial_\xi^2 L(0) = 0,\ \partial_{\xi\xi'}^2 L(0) = 0,\ \partial_{\xi'}^2 L(0) \neq 0,\ \partial_\xi^3 L(0) \neq 0;$$

clearly L satisfies the classical transversality condition and in consequence $\mathcal{L}$ satisfies this condition as well. The Malgrange preparation theorem suggests that in suitable local coordinates we could take

$$L(x, x', \xi, \xi') = \xi^3 + \xi'^2 + f(x, x', \xi')\xi + g(x, x')$$

with f, g smooth functions and the corresponding Legendre transformation is

$$\mathfrak{L}(x, x', \xi, \xi') = (x, x', 3\xi^2 + f(x, x', \xi'), \xi\partial_{\xi'} f(x, x', \xi') + 2\xi').$$

The energy function is

$$\mathcal{E}_{\rm L}(x,x',\xi,\xi') = \xi\partial_\xi L(x,x',\xi,\xi') + \xi'\partial_{\xi'} L(x,x',\xi,\xi') - L(x,x',\xi,\xi')$$

and the pullback of the canonical symplectic form by the Legendre transform is

$$\mathcal{L}^*\Omega(x,x',\xi,\xi') = 6\xi\,{\rm d}\xi\wedge{\rm d}x + 2\,{\rm d}\xi'\wedge{\rm d}x' + {\rm d}f(x,x',\xi')\wedge{\rm d}x$$
$$+\xi\,{\rm d}(\partial_{\xi''} f(x,x',\xi'))\wedge{\rm d}x' + \partial_{\xi''} f(x,x',\xi')\,{\rm d}\xi\wedge{\rm d}x'.$$

If we require for simplicity that the function f does not depend on ξ', then the critical locus is $\{\xi = 0\}$. The kernel's condition on the critical locus is written as

$$({\rm d}f\wedge{\rm d}x + 2\,{\rm d}\xi'\wedge{\rm d}x')\wedge(\xi'\,{\rm d}\xi') = 0$$

and this implies that a depends only on x and the energy function written as

$$\mathcal{E}_{\rm L}(x,x',\xi,\xi') = 2\xi^3 + \xi'^2 + f(x)\xi - g(x').$$

Then the Lagrangian vector field is

$$X_{\rm L}(x,x',\xi,\xi') = \xi\frac{\partial}{\partial x} + \xi'\frac{\partial}{\partial x'} - (f'(x)/6)\frac{\partial}{\partial\xi} - (g'(x')/2)\frac{\partial}{\partial\xi'}$$

and the projection of its integral curves gives the solution of this problem on $\mathbb{R}^2$; we observe that, near the smooth part of the critical locus, the Legendre transform is a folding map and the flow decouples into two independent regular flows in the (x,ξ) and (x,ξ') planes, cf. [27].

4. Another approach to the genericity of quadratic forms

4.1. Singularities of the pullback of a pseudo-Riemannian metric

Let $(\mathcal{N},\mathfrak{g})$ be an n-dimensional pseudo-Riemannian manifold and we consider an equidimensional manifold $\mathcal{M}$. If $\mathfrak{h}:\mathcal{M}\to\mathcal{N}$ is a smooth map, we will turn our attention to the consequences of the appearance of the degeneracy locus of the pullback $\mathfrak{h}^*\mathfrak{g}$ on $\mathcal{M}$. The kernel of $\mathfrak{h}^*\mathfrak{g}$ at $x\in\mathcal{M}$ is

$${\rm Ker}_x\mathfrak{h}^*\mathfrak{g} = \{v\in T_x\mathcal{M}/D_x\mathfrak{h}(v)\in {\rm Im}\,D_x\mathfrak{h}\cap{\rm Im}\,D_x\mathfrak{h}^\perp\},$$

where ${\rm Im}\,D_x\mathfrak{h}^\perp$ denotes the orthogonal complement of ${\rm Im}\,D_x\mathfrak{h}$ in the quadratic space $T_{\mathfrak{h}(x)}\mathcal{N}$ and obviously

$${\rm Ker}\,D_x\mathfrak{h}\subseteq{\rm Ker}_x\mathfrak{h}^*\mathfrak{g}.$$

The vector bundle ${\rm Hom}(\pi_1^*T\mathcal{M},\pi_2^*T\mathcal{N})$ where π_1 and π_2 are the projections of the product $\mathcal{M}\times\mathcal{N}$ on the first and second factor, respectively, carries its classical stratification by the rank, cf. [3]. However, we will construct a refinement of this stratification that encounters the behavior of pullbacks of $\mathfrak{g}$ on the fibers of $T\mathcal{N}$. Indeed, the structural group of the vector bundle $T\mathcal{N}$ is reduced to the group $\mathbf{O}(\mathfrak{g})$ of linear transformations that preserve in every fiber the pseudo-Riemannian metric and the stratification of the fiber type $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ comes from the action of the group $GL(n, \mathbb{R}) \times \mathbf{O}(\mathfrak{g})$ by left–right multiplication.

Lemma 7. *If* $\mathfrak{g}$ *is a pseudo-Riemannian metric on* $\mathbb{R}^n$*, then the orbits* $\Sigma_{k,\varsigma}$ *in* $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ *of the action of* $GL(n, \mathbb{R}) \times \mathbf{O}(\mathfrak{g})$ *by left–right multiplication are classified by the numbers:*

$$\dim \mathrm{Ker}\, \mathfrak{l} = k \quad \text{and} \quad \mathrm{sgn}(\mathfrak{g}|\mathrm{Im}\, \mathfrak{l}) = (p, q, s) = \varsigma,$$

where $\mathfrak{l} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ *and*

$$\mathrm{codim}\, \Sigma_{k,\varsigma} = k^2 + s(s+1)/2.$$

Proof. The space

$$M_r = \{(\mathfrak{l}, P) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) \times \mathbf{G}_{n,r}(\mathbb{R}) / \mathrm{Im}\, \mathfrak{l} \subseteq P\}$$

after projection on the second factor is exhibited as an algebraic vector bundle over the Grassmannian $\mathbf{G}_{n,r}(\mathbb{R})$ of rank rn; the fiber over $P \in \mathbf{G}_{n,r}(\mathbb{R})$ consists of the linear maps with their image contained in P. The manifold Σ^r of linear maps of rank r is fibered by this projection on the first factor, by those linear maps with image exactly P. Furthermore, the r-dimensional subspaces E of $\mathbb{R}^n$ such that $\mathrm{sgn}(\mathfrak{g}|E) = (p, q, s)$ form semialgebraic subvarieties of $\mathbf{G}_{n,r}(\mathbb{R})$. These subspaces are tangent to the quadric $\{\mathfrak{g} = 0\}$ along a s'-dimensional linear subspace and since the quadric is an orbit of $\mathbf{O}(\mathfrak{g})$, the group acts transitively on the variety of these planes; therefore these constitute a smooth semialgebraic subvariety of codimension $s(s+1)/2$. On the other hand, considering the fibers over $\mathbf{G}_{n,r}(\mathbb{R})$ of Σ^r, we observe that over the planes P with $\mathrm{Ker}(\mathfrak{g}|P) = s$ the maps $\mathfrak{l} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ in the fiber above that correspond to the orbit are those for which $(\mathrm{Ker}\, \mathfrak{l}) \cap \mathrm{Ker}(\mathfrak{g}|\mathrm{Im}\, \mathfrak{l}) = \{0\}$. Hence, the linear maps of the given rank that have their image in such sets constitute the orbits of the action of $GL(n, \mathbb{R}) \times \mathbf{O}(\mathfrak{g})$. We conclude then finally that the codimension of the orbit $\Sigma_{k,\varsigma}$ of $\mathfrak{l} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ with $\dim \mathrm{Ker}\, \mathfrak{l} = n - r = k$ and $\mathrm{sgn}(\mathfrak{g}|\mathrm{Im}\, \mathfrak{l}) = (p, q, s) = \varsigma$ is $k^2 + s(s+1)/2$. □

The action of the reduced structural group transfers the above stratification of each fiber to the vector bundle $\mathrm{Hom}(\pi_1^* T\mathcal{M}, \pi_2^* T\mathcal{N})$ and hence it is lifted to a stratification of the space $J^1(\mathcal{M}, \mathcal{N})$ of 1-jets of the elements of $C^\infty(\mathcal{M}, \mathcal{N})$. Therefore, the smooth maps $\mathfrak{h} : \mathcal{M} \to \mathcal{N}$ with their differential transverse to this stratification constitute an open and dense set in the Baire space $C^\infty(\mathcal{M}, \mathcal{N})$; in the sequel they will be called *generic*. We remark here that despite the fact of $\mathfrak{h} : \mathcal{M} \to \mathcal{N}$ being a generic map, the quadratic form $\mathfrak{h}^*\mathfrak{g}$ is never generic in the sense of Section 1.1. If $\mathfrak{h}$ is a generic map, then the manifold $\mathcal{M}$ inherits a coherent stratification where every stratum

$$\Sigma_{k,s}(\mathfrak{h}^*\mathfrak{g}) = \{x \in \mathcal{M} / \dim \mathrm{Ker}_x\, D\mathfrak{h} = k, \dim \mathrm{Ker}_x\, \mathfrak{h}^*\mathfrak{g} = s\}, \quad k, s \in \mathbb{N},$$

is, if nonempty, smooth submanifold of codimension $k^2 + s(s+1)/2$. The regular locus of $\mathfrak{h}^*\mathfrak{g}$ is the open and dense set $\Sigma_{0,0}(\mathfrak{h}^*\mathfrak{g})$ and the degeneracy locus

$$\Sigma(\mathfrak{h}^*\mathfrak{g}) = \bigcup_{(k,s)\neq(0,0)} \Sigma_{k,s}(\mathfrak{h}^*\mathfrak{g})$$

is a singular hypersurface with the smooth part containing the open and dense set $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$; the strata $\Sigma_{1,1}(\mathfrak{h}^*\mathfrak{g})$, appearing when $\mathfrak{g}$ is not positive define, is of codimension 2 in $\mathcal{M}$. Furthermore, following the approach of Section 3.2, we could construct the generic model near a point of $\Sigma_{k,s}(\mathfrak{h}^*\mathfrak{g})$. We conclude that $\Sigma(\mathfrak{h}^*\mathfrak{g})$ is locally diffeomorphic to a determinantal variety defined by the zeros of a quadratic-affine polynomial that satisfy the assumption of Lemma 1 and hence possesses the division property in the ring of germs of smooth functions on $\mathbb{R}^n$.

4.2. The quadratic morphism and the second order kernel condition

Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and we consider a equidimensional manifold $\mathcal{M}$. If $\mathfrak{h} : \mathcal{M} \to \mathcal{N}$ is a generic map, in order to treat the $\mathfrak{h}^*\mathfrak{g}$-morphism over $\mathcal{M}$, we consider the ideal $\mathfrak{I}_x(\Sigma(\mathfrak{h}^*\mathfrak{g}))$ of germs at $x \in \Sigma(\mathfrak{h}^*\mathfrak{g})$ of smooth functions that vanish identically on $\Sigma(\mathfrak{h}^*\mathfrak{g})$ in the ring of germs of smooth function on $\mathcal{M}$. A pfaffian form $\mathfrak{b}$ on $\mathcal{M}$ is said to satisfy the *second order kernel condition* at the points $x \in \Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ if, for every germ of vector field $\mathcal{X}$ such that $\mathcal{X}(x) \in \mathrm{Ker}_x\, \mathfrak{h}^*\mathfrak{g}$, there holds

$$\mathfrak{b}(\mathcal{X})(x) \in (\mathfrak{I}_x(\Sigma(\mathfrak{h}^*\mathfrak{g})))^2.$$

Theorem 8. *If $(\mathcal{N}, \mathfrak{g})$ is a pseudo-Riemannian manifold and $\mathfrak{h} : \mathcal{M} \to \mathcal{N}$ a generic map from an equidimensional manifold $\mathcal{M}$, then the pfaffian forms which possess a gradient vector field defined everywhere on $\mathcal{M}$ are exactly these which satisfy the second order kernel condition at the points of $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$.*

Proof. Let $\mathfrak{b}$ be a differential form on $\mathcal{M}$ and $\mathcal{X}_\mathfrak{b}$ the vector field defined out of the degeneracy locus by the equation

$$(\mathfrak{h}^*\mathfrak{g})(\mathcal{X}_\mathfrak{b}, .) = \mathfrak{b}(.).$$

In the neighborhood of a point of $\Sigma(\mathfrak{h}^*\mathfrak{g})$, we trivilize the bundles, and in the local matricial context, using the same notations, we look at the corresponding matricial equation

$$(\mathfrak{h}^*\mathfrak{g}(x))\mathcal{X}_\mathfrak{b}(x) = \mathfrak{b}(x).$$

We stand at a point $a \in \Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ and following the approach of Section 3.2, using the classical methods of universal unfolding, we observe that there exist germs of a diffeomorphism and smooth map, respectively,

$$\varphi : (\mathbb{R}^n, 0) \to (\mathcal{M}, a) \quad \text{and} \quad (\psi\psi') : (\mathbb{R}^n, 0) \to GL(n, \mathbb{R}) \times \mathbf{O}(\mathfrak{g})$$

such that, in the neighborhood of $0 \in \mathbb{R}^n$, the induced quadratic form suffers the change

$$\mathfrak{h}^*\mathfrak{g}(\varphi(x)) = {}^t(\psi'(x)\Xi(x)\psi(x))\mathfrak{g}(\mathfrak{h}\circ\varphi(x))(\psi'(x)\Xi(x)\psi(x)),$$

where

$$\Xi(x) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & x_n \end{pmatrix}.$$

Always, with the same notations, the above equation is written as

$${}^t\Xi(x){}^t\psi'(x)\mathfrak{g}(\mathfrak{h}\circ\varphi(x))\psi'(x)\Xi(x)X(x) = B(x)$$

with

$$X(x) = \psi(x)(D\varphi^{-1})(\varphi(x))\mathcal{X}_{\dagger}(\varphi(x))$$

and

$$B(x) = {}^t(\psi(x)((D\varphi^{-1})(\varphi(x)))^{-1}\mathfrak{b}(\varphi(x)).$$

The matrix of the quadratic form ${}^t\psi'(x)\mathfrak{g}(\mathfrak{h}\circ\varphi(x))\psi'(x)$ is written as

$$\begin{pmatrix} A(x) & C(x) \\ {}^tC(x) & D(x) \end{pmatrix},$$

where $A : (\mathbb{R}^n, 0) \to \mathcal{S}^2(\mathbb{R}^{n-1})^*$, $C : (\mathbb{R}^n, 0) \to \mathbb{R}^{n-1}$, $D : (\mathbb{R}^n, 0) \to \mathbb{R}$ are smooth maps and we are led to the following system of equations:

$$A(x)X'(x) + x_nC(x)X_n(x) = B'(x)$$

and

$$x_n{}^tC(x)X'(x) + x_n^2D(x)X_n(x) = B_n'(x),$$

where (X', X_n) and (B', B_n) account for the obvious splitting of X and B, respectively, into their component functions; as we verify by a direct calculation with the normal form, at points of $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ that the matrix $A(x)$ is invertible. Therefore, we obtain the following equation:

$$B'(x_nA(x)^{-1}C(x)) - B_n'(x)x_n^2(\langle C(x), A(x)^{-1}C(x)\rangle + D(x)) = 0,$$

where $\langle\, ,\rangle$ stands for the usual Euclidean inner product. Clearly, these equations admit a formal solution if and only if $\mathfrak{b}$ satisfies the second order condition on $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$. Indeed, all the vector fields that enter the kernel of $\mathfrak{h}^*\mathfrak{g}$ at points of $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ are written in the context of the normal form as $(x_kA(x)^{-1}X)$, χ, (X, χ) standing for the components of a smooth vector field. Notice that the second order kernel condition is expressed through the last equation. Hence, we obtain a local smooth solution on $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$. Now, stand at an arbitrary point of $\Sigma(\mathfrak{h}^*\mathfrak{g})$ and return to the initial matricial equation that obviously is written as

$$\det \mathfrak{h}^*\mathfrak{g}(x)\mathcal{X}_{\mathfrak{b}}(x) = {}^{c}(\mathfrak{h}^*\mathfrak{g}(x))\mathfrak{b}(x)$$

or that

$$(\det D\mathfrak{h}(x))^2\mathcal{X}_{\mathfrak{b}}(x) = {}^{c}(\mathfrak{h}^*\mathfrak{g})(x)\mathfrak{b}(x).$$

The existence of a solution of this equation on $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ implies the vanishing of the second member on $\Sigma_{1,0}(\mathfrak{h}^*\mathfrak{g})$ and by continuity on $\Sigma(\mathfrak{h}^*\mathfrak{g})$. After Section 4.1, in suitable coordinates, the coefficient of the first member of this equation is the square of a quadratic-affine polynomial possessing the division property since it fulfills the assumption of Lemma 1. This holds for the second member of the above equation provided we perform successively twice the division process in the ring of germs of smooth functions. Therefore, we achieve a local smooth solution and with a partition of unity a global smooth solution on $\mathcal{M}$. □

Let $(\mathcal{N}, \mathfrak{g})$ be a pseudo-Riemannian manifold and $\mathfrak{h} : \mathcal{M} \to \mathcal{N}$ a generic map from a equidimensional manifold $\mathcal{M}$ equipped with the quadratic form $\mathfrak{q} = \mathfrak{h}^*\mathfrak{g}$. The degeneracy locus $\Sigma(\mathfrak{q})$ coincides with the projection of the degeneracy locus

$$\Sigma(\Omega_{\mathfrak{q}}) = \{(x, y) \in T\mathcal{M}/\mathrm{Ker}_{(x,y)}\Omega_{\mathfrak{q}} \neq \{0\}\}$$

of the Lagrangian form $\Omega_{\mathfrak{q}}$ as the pullback of the standard symplectic form Ω on $T^*\mathcal{M}$ through the $\mathfrak{q}$-morphism over $\mathcal{M}$. We consider the energy function defined on the tangent bundle by

$$\mathcal{E}_{\mathfrak{q}}(\mathfrak{v}_x) = \tfrac{1}{2}\mathfrak{q}(x)(\mathfrak{v}_x, \mathfrak{v}_x) + V(\pi(\mathfrak{v}_x)), \quad x \in \mathcal{M},$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ denotes the canonical projection and V is a potential function on $\mathcal{M}$. The Lagrangian vector field $\mathcal{X}_{\mathfrak{q}}$ associated to $\mathcal{E}_{\mathfrak{q}}$ is defined, out of $\Sigma(\Omega_{\mathfrak{q}})$, by the equation

$$\Omega_{\mathfrak{q}}(\mathcal{X}_{\mathfrak{q}}, .) = -\,\mathrm{d}\mathcal{E}_{\mathfrak{q}}(.).$$

The projection of the integral curves of $\mathcal{X}_{\mathfrak{q}}$ contained in the complement of $\Sigma(\Omega_{\mathfrak{q}})$ gives the geodesics relative to $\mathfrak{q}$ in the complement of $\Sigma(\mathfrak{q})$ in $\mathcal{M}$. The Lagrangian form $\Omega_{\mathfrak{q}}$ is not generic in the sense that it is not transverse to the stratification of the vector bundle of the antisymmetric forms on $T\mathcal{M}$. The Lagrangian vector field $\mathcal{X}_{\mathfrak{q}}$ cannot be extended smoothly on the whole $T\mathcal{M}$; the differential of the energy function does not annilhate everywhere the kernel of $\Omega_{\mathfrak{q}}$. Using the generic local model of Section 4.1, a model for $\Omega_{\mathfrak{q}}$ can be constructed, and by a direct calculation, we are led to the determination of a subset of $\Sigma(\Omega_{\mathfrak{q}})$ on which the extension criterion is satisfied and this part constitutes the frontier up to which the $\mathcal{X}_{\mathfrak{q}}$ admits a smooth extension. Also, when a curve $\gamma : [a, b] \to \mathcal{M}$ is contained in the complement of $\Sigma(\mathfrak{q})$, then $(\gamma, \dot{\gamma}) : [\alpha, \beta] \to T\mathcal{M}$ is an integral curve of $\mathcal{X}_{\mathfrak{q}}$ if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, where ∇ denotes the Lévi-Civitã connection associated with $\mathfrak{q}$ out of $\Sigma(\mathfrak{q})$. A singular connection constructed according to the approach of Section 2.3 can be used for the determination of the singular geodesics of the quadratic structure $(\mathcal{M}, \mathfrak{q})$.

4.3. Comments on the wave front of the fundamental solution of second order differential operators

Consider the second order differential operator

$$\mathcal{P} = \sum_{1\le i\le j\le n} a_{ij}(x)\partial_i\partial_j + \sum_{1\le i\le n} b_i(x)\partial_i + c(x)$$

with a_{ij}, b_i, c smooth functions on $\mathbb{R}^n$ and the linear equation thus defined

$$\mathcal{P}u = f$$

with f element of the Sobolev space

$$\mathcal{H}^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n)/(1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of temperate distributions on $\mathbb{R}^n$ and $\hat{u}$ the Fourier transform of u. According to a classical theorem of Hörmander, cf. [13], the wave front set of the fundamental solution $\mathcal{WF}(u)$ is determined by the characteristic variety

$$\operatorname{char}\mathcal{P} = \left\{(x,\xi) \in T^*\mathbb{R}^n - \{0\}/P(x,\xi) = \sum_{1\le i\le j\le n} .a_{ij}(x)\xi_i\xi_j = 0\right\}.$$

Precisely the theorem states that if $\gamma : [t_0, t_1] \to T^*\mathbb{R}^n - \{0\}$ is a null bicharacteristic strip for P and $u \in \mathcal{H}^{s+1}$ at $\gamma(t_1)$, then $u \in \mathcal{H}^{s+1}$ on γ. Furthemore,

$$\mathcal{WF}(u) - \mathcal{WF}(f) \subset \operatorname{char}\mathcal{P}$$

and is invariant under the flow of the Hamiltonian vector field that corresponds to P:

$$H_P(x,\xi) = \sum_{1\le i\le j\le n}\left(\sum_{1\le j\le n} a_{ij}(x)\xi_i\frac{\partial}{\partial x_j} - \sum_{1\le k,l\le n}\frac{\partial P}{\partial x_j}(x,\xi)\frac{\partial}{\partial \xi_j}\right).$$

Let $\mathfrak{q} : \mathbb{R}^n \to \mathcal{S}^2(T^*\mathbb{R}^n)$ be the quadratic form defined by

$$\mathfrak{q}(x)(\xi,\xi) = P(x,\xi).$$

The preceeding theory of quadratic forms allows us to comment on the wave front set of a generic second order equation in the case where $\mathfrak{q}$ is a generic quadratic form or the generic restriction of a regular quadratic form or the generic pullback of a regular quadratic form. Precisely, the lemma of related germs, cf. [29], allows to obtain the generic normal form of such an equation: in suitable local coordinates

$$x = \varphi(\bar{x}) \quad \text{and} \quad \xi = \mathfrak{g}(\bar{x})\bar{\xi},$$

for a diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a smooth map $\psi : (\mathbb{R}^n, 0) \to GL(n, \mathbb{R})$, we have that

$$P(\bar{x},\bar{\xi}) = \sum_{1\le i\le j\le n} \varepsilon_i\bar{\xi}_i^2 + \sum_{1\le i\le n} \bar{x}_{\tau(i,j)}\bar{\xi}_i\bar{\xi}_j.$$

An easy computation shows that the characteristic variety is nonsingular in $T^*\mathbb{R}^n - \{0\}$; indeed the singular set is contained in the zero section of $T^*\mathbb{R}^n$. Furthemore, the degeneracy locus of $\mathfrak{q}$ gives rise to the critical set

$$\mathfrak{m}(\mathcal{P}) = \{(x, \xi) \in \text{char}\,\mathcal{P} / x \in \Sigma(\mathfrak{q}),\ \xi \in \text{Ker}_x \mathfrak{q}\}.$$

The preceeding theorem on the singularities of differential operators gives an interpretation to this critical set. Indeed if $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection, then

$$\pi(\mathfrak{m}(\mathcal{P})) \subset \text{sing}(\text{supp}(u)).$$

An interesting issue, thus, raises in the study of the asymptotic behavior of the fundamental solution through the Lax–Ludwig method [6,9] as we approach this set, in the context of the results obtained in [27].

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